

MOTION IN TWO AND THREE DIMENSIONS

In this chapter we consider an extension of the concepts presented in Chapters 2 and 3. In those chapters we introduced kinematics and dynamics in terms of vectors, but we considered only applications in one dimension. In this chapter we broaden the discussion to include two- and three-dimensional applications. Keeping track of the separate x , y , and z components of the motion is greatly simplified if we rely on vectors to describe the particle's position, velocity, and acceleration, as well as the forces that may act on the particle. To illustrate the vector techniques, we discuss two examples: a projectile launched with both horizontal and vertical velocity components in the Earth's gravity, and an object moving in a circular path.

4-1 MOTION IN THREE DIMENSIONS WITH CONSTANT ACCELERATION

In Section 2-5 we developed a procedure for analyzing the position, velocity, and acceleration of a particle that moves in one dimension with constant acceleration. Knowing the acceleration, we can find the velocity at all times according to Eq. 2-26 ($v_x = v_{0x} + a_x t$) and the position at all times from Eq. 2-28 ($x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$).

Now we consider the possibility that the particle moves in three dimensions with constant acceleration. That is, as the particle moves, the acceleration does not vary in either magnitude or direction. Equivalently, we can represent the acceleration as a vector $\vec{a}$ with three components (a_x, a_y, a_z), each of which is constant. In general the particle moves in a curved path. As is the case in one-dimensional motion, we would like to know the particle's velocity $\vec{v}$ (a vector with components v_x, v_y, v_z) and its position $\vec{r}$ (a vector with components x, y, z) at all times.

We can obtain the general equations for motion with constant $\vec{a}$ by setting

$$a_x = \text{constant}, \quad a_y = \text{constant}, \quad \text{and} \quad a_z = \text{constant}.$$

The particle starts at $t = 0$ with initial position $\vec{r}_0 = x_0 \hat{i} + y_0 \hat{j} + z_0 \hat{k}$ and an initial velocity $\vec{v}_0 = v_{0x} \hat{i} + v_{0y} \hat{j} + v_{0z} \hat{k}$. We now proceed as we did in Section 2-5 and develop, in analogy with Eq. 2-26, *three* one-dimensional equations: $v_x = v_{0x} + a_x t$, $v_y = v_{0y} + a_y t$, and $v_z = v_{0z} + a_z t$, which we write as the single three-dimensional vector equation

$$\vec{v} = \vec{v}_0 + \vec{a}t. \quad (4-1)$$

When using this or any other vector equation, remember that it represents three independent one-dimensional equations. That is, a vector equality such as $\vec{A} = \vec{B}$ means that three conditions *must* be fulfilled: $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$. In this way it is clear how Eq. 4-1 represents the three one-dimensional equations for the components.

By referring to the three component equations as “independent,” we mean that the velocity components vary independently of one another—for example, a_x affects only v_x and not v_y or v_z . If $a_y = a_z = 0$ but $a_x \neq 0$, then v_y and v_z would remain constant but v_x would vary with time.

The second term on the right side of Eq. 4-1 involves the multiplication of the vector $\vec{a}$ by the scalar t . As discussed in Appendix H, this gives a vector of length at that points in the same direction as the original vector $\vec{a}$.

In a similar way, we can write the three equations for the components of the position vector, as in Eq. 2-28: $x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$, $y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$, $z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$. These three one-dimensional equations can be combined into a single three-dimensional vector equation:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2, \quad (4-2)$$

which contains within it the three one-dimensional equations for the components.

SAMPLE PROBLEM 4-1. Starship *Enterprise* is coasting through space (where gravity is negligible) at a speed of 15.0 km/s relative to a particular inertial reference frame. Suddenly, the ship is gripped by a tractor beam, which pulls it in a direction perpendicular to its original velocity and gives it an acceleration of 4.2 km/s² in that direction. After the tractor beam has acted for 4.0 s, the *Enterprise* fires its engines, giving the ship a constant acceleration of 18.0 km/s² in the direction parallel to its original motion. After an additional 3.0 s, both the engines and the tractor beam stop operating, and the ship is again coasting. Find the ship's velocity at that time and its position relative to its location when the tractor beam first appeared.

Solution We set up our coordinate system with the positive x axis in the direction of the ship's original motion and the positive y axis in the direction of the tractor beam's pull, and we choose the origin ($x = 0, y = 0$) at the location where the tractor beam began to act. (With this choice of coordinate system, there is no motion in the z direction.) The problem breaks into two parts, which must be analyzed separately: (1) from $t = 0$ to $t = 4.0$ s the ship moves with $a_x = 0$, $a_y = +4.2$ km/s², and (2) from $t = 4.0$ s to $t = 7.0$ s it moves with $a_x = +18.0$ km/s², $a_y = +4.2$ km/s².

We analyze each part in turn. For the first part, with $v_{0x} = +15.0$ km/s and $v_{0y} = 0$, the x and y components of Eqs. 4-1 and 4-2 become

$$\begin{aligned} v_x &= v_{0x} + a_x t = 15.0 \text{ km/s} + 0 = 15.0 \text{ km/s} \\ v_y &= v_{0y} + a_y t = 0 + (4.2 \text{ km/s}^2)(4.0 \text{ s}) = 16.8 \text{ km/s} \\ x &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 = 0 + (15.0 \text{ km/s})(4.0 \text{ s}) + 0 = 60.0 \text{ km} \\ y &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = 0 + 0 + \frac{1}{2}(4.2 \text{ km/s}^2)(4.0 \text{ s})^2 \\ &= 33.6 \text{ km}. \end{aligned}$$

For the 3.0-s interval from $t = 4.0$ s to $t = 7.0$ s, we write a similar set of equations using a new variable t' that ranges from 0 to 3.0 s (but we keep the origin of the coordinate system in the original location). For this interval, the starting velocities and locations are the values found above for $t = 4.0$ s ($v_{0x} = 15.0$ km/s, $v_{0y} = 16.8$ km/s, $x_0 = 60.0$ km, $y_0 = 33.6$ km), and so

$$\begin{aligned} v_x &= v_{0x} + a_x t' = 15.0 \text{ km/s} + (18.0 \text{ km/s}^2)(3.0 \text{ s}) = 69.0 \text{ km/s} \\ v_y &= v_{0y} + a_y t' = 16.8 \text{ km/s} + (4.2 \text{ km/s}^2)(3.0 \text{ s}) = 29.4 \text{ km/s} \\ x &= x_0 + v_{0x}t' + \frac{1}{2}a_x t'^2 \\ &= 60.0 \text{ km} + (15.0 \text{ km/s})(3.0 \text{ s}) + \frac{1}{2}(18.0 \text{ km/s}^2)(3.0 \text{ s})^2 \\ &= 186 \text{ km} \\ y &= y_0 + v_{0y}t' + \frac{1}{2}a_y t'^2 \\ &= 33.6 \text{ km} + (16.8 \text{ km/s})(3.0 \text{ s}) + \frac{1}{2}(4.2 \text{ km/s}^2)(3.0 \text{ s})^2 \\ &= 103 \text{ km}. \end{aligned}$$

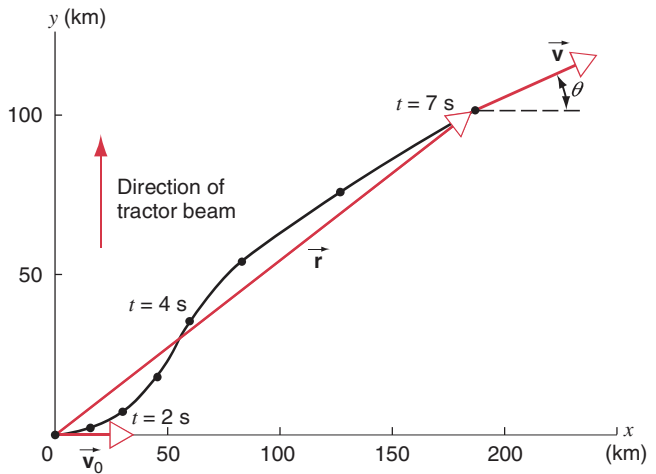


FIGURE 4-1. Sample Problem 4-1. The dots show the position of the starship at successive 1-second intervals from $t = 0$ to $t = 7$ s. The vectors $\vec{r}$ and $\vec{v}$ show the position and velocity at $t = 7$ s. Note that $\vec{v}_0$ is tangent to the path at $t = 0$, and $\vec{v}$ is tangent to the path at $t = 7$ s.

At $t = 7.0$ s, the ship is at $x = 186$ km, $y = 103$ km (or, a distance $r = \sqrt{x^2 + y^2} = 213$ km from the initial reference point). The components of its velocity are $v_x = 69$ m/s, $v_y = 29$ m/s, corresponding to a speed $v = \sqrt{v_x^2 + v_y^2} = 75$ km/s and a direction given by the angle $\theta = \tan^{-1} v_y/v_x = 23^\circ$ relative to the x axis. Figure 4-1 shows the path of the starship and its position at various times. In general, as shown at $t = 7$ s, the directions of the position and velocity vectors are different.

Note in this problem how using the components of vectors helps us to separate the x and y motions. That is, the solutions to the equations for the x components do not depend on the y motion. If the tractor beam did not exist and the *Enterprise* fired its engines in the same way from 4.0 s to 7.0 s, it would still be at $x = 186$ km moving with velocity $v_x = 69$ km/s at 7.0 s. If the tractor beam were present but the engines did not fire, the *Enterprise* would still be at $y = 103$ km moving with $v_y = 29$ km/s at 7.0 s.

4-2 NEWTON'S LAWS IN THREE-DIMENSIONAL VECTOR FORM

Before we can write Newton's laws in their three-dimensional vector form, we first must verify that force, as we have defined it, is a vector quantity. We have seen in Chapter 3 that, even in one dimension, we must carefully account for the direction of each force. Since force has both magnitude and direction, we suspect that it may be a vector quantity. However, to be a vector it is not enough for a quantity to have both magnitude and direction; it also must obey the laws of vector addition described in Section 2-2. Only from experiment can we learn whether forces, as we defined them, obey these laws.

Let us arrange to exert a force of 4 N along the x axis and a force of 3 N along the y axis. We apply these forces first separately and then simultaneously to the standard 1-kg body placed, as before, on a horizontal, frictionless sur-

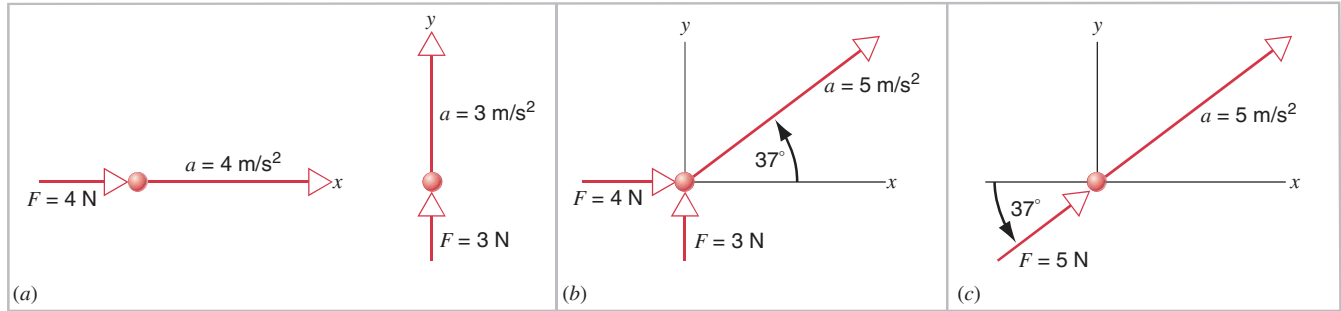


FIGURE 4-2. (a) A 4-N force in the x direction gives an acceleration of 4 m/s^2 in the x direction, and a 3-N force in the y direction gives an acceleration of 3 m/s^2 in the y direction. (b) When the forces are applied simultaneously, the resultant acceleration is 5 m/s^2 in the direction shown. (c) The same acceleration can be produced by a single 5-N force in the direction shown.

face. What will be the acceleration of the standard body? We would find by experiment that the 4-N force in the x direction acting alone produced an acceleration of 4 m/s^2 in the x direction, and that the 3-N force in the y direction acting alone produced an acceleration of 3 m/s^2 in the y direction (Fig. 4-2a). When the forces are applied simultaneously, as shown in Fig. 4-2b, we find that the acceleration is 5 m/s^2 directed along a line that makes an angle of 37° with the x axis. This is the same acceleration that would be produced if the standard body were experiencing a force of 5 N in that direction. This same result can be obtained if we first add the 4-N and 3-N forces vectorially (Fig. 4-2c) to a 5-N resultant directed at 37° from the x axis, and then apply that single 5-N net force to the body. Experiments of this kind show conclusively that forces are vectors; they have magnitude and direction, *and* they add according to the vector addition law.

Now that we are convinced that force is a vector, we are justified in writing Newton's second law in vector form, as we already have done in Chapter 3:

$$\sum \vec{F} = m\vec{a} \quad (4-3)$$

which includes the three component equations

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z. \quad (4-4)$$

That is, all three equations must simultaneously be satisfied when we apply Newton's second law.

Equation 4-3 suggests that we can find the direction of the acceleration by taking the vector sum of all the forces acting on the particle. Because the mass m is a scalar, the direction of $\vec{a}$ is the same as the direction of $\sum \vec{F}$. Once we have the resultant force we can also find the magnitude of $\vec{a}$ from Eq. 4-3. However, as we shall see in the examples in this chapter, it is often easier to use Eqs. 4-4 to solve problems by first resolving each force into its components and then finding the sum of each of the force components to obtain each component of the acceleration.

If the forces are constant, then the acceleration is constant and we can use the equations of Section 4-1 to find the position and velocity of the particle at all times. If the forces are not constant, then it is not possible to use the equations for constant acceleration; an example of a nonconstant force, the drag on a projectile, is discussed later in this chapter.

Newton's third law is also a vector equation:

$$\vec{F}_{AB} = -\vec{F}_{BA}, \quad (4-5)$$

which tells us that, no matter what the direction in three-dimensional space of the vector $\vec{F}_{AB}$ that represents the force exerted on A by B, the vector $\vec{F}_{BA}$ that represents the force on B by A has the same magnitude and acts in the opposite direction.

SAMPLE PROBLEM 4-2. A crate of mass $m = 62 \text{ kg}$ is sliding without friction with an initial velocity of $v_0 = 6.4 \text{ m/s}$ along the floor. In an attempt to move it in a different direction, Tom pushes opposite to its initial motion with a constant force of magnitude $F_{CT} = 81 \text{ N}$, while Jane pushes in a perpendicular direction with a constant force of magnitude $F_{CJ} = 105 \text{ N}$ (Fig. 4-3a). If they each push for 3.0 s, in what direction is the crate moving when they stop pushing?

Solution Let us take the positive x direction to be that of the initial motion of the crate (so that Tom's force $\vec{F}_{CT}$ is in the negative x direction) and the positive y direction to be that of Jane's force $\vec{F}_{CJ}$. Figure 4-3b shows the free-body diagram of the crate. The only force in the x direction is that exerted by Tom, so $\sum F_x = -F_{CT}$; similarly, $\sum F_y = F_{CJ}$. Using Newton's second law in its component form (Eqs. 4-4) we can write the equations of motion of the crate as

$$x \text{ direction } (\sum F_x = ma_x): \quad -F_{CT} = ma_x$$

$$y \text{ direction } (\sum F_y = ma_y): \quad -F_{CJ} = ma_y$$

Solving, we find

$$a_x = -\frac{F_{CT}}{m} = \frac{-81 \text{ N}}{62 \text{ kg}} = -1.31 \text{ m/s}^2$$

$$a_y = \frac{F_{CJ}}{m} = \frac{105 \text{ N}}{62 \text{ kg}} = 1.69 \text{ m/s}^2.$$

Using Eq. 4-1 in its component form, we can find the velocity components at $t = 3.0 \text{ s}$:

$$v_x = v_{0x} + a_x t = 6.4 \text{ m/s} + (-1.31 \text{ m/s}^2)(3.0 \text{ s}) = 2.5 \text{ m/s}$$

$$v_y = v_{0y} + a_y t = 0 + (1.69 \text{ m/s}^2)(3.0 \text{ s}) = 5.1 \text{ m/s}.$$

A graph of the path of the crate is shown in Fig. 4-3c, which also shows the velocity components at $t = 3.0 \text{ s}$. To find the magnitude

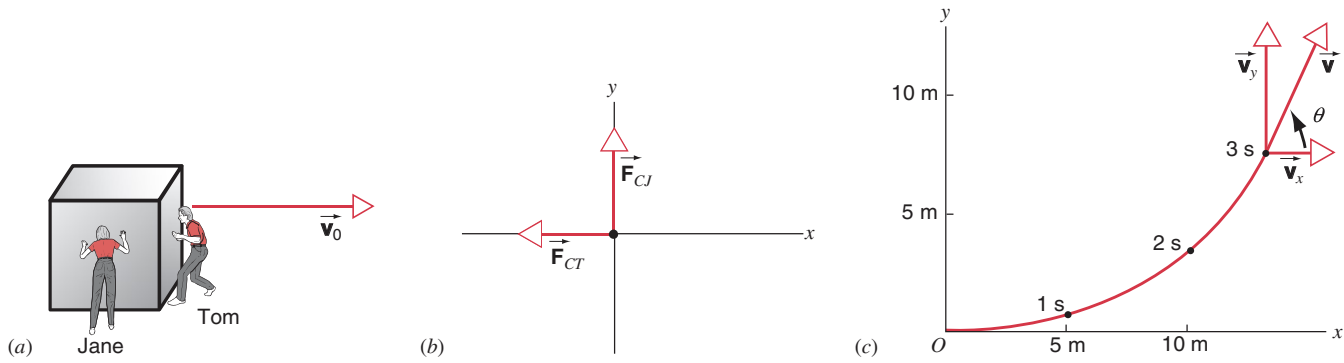


FIGURE 4-3. Sample Problem 4-2. (a) Tom pushes opposite to the initial motion of the crate and Jane pushes in a perpendicular direction. (b) The free-body diagram of the crate. (c) The tangent to the path gives the direction of the crate's motion. The velocity components at $t = 3.0$ s are shown.

of the velocity and its direction, we use Eqs. 2-2:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2.5 \text{ m/s})^2 + (5.1 \text{ m/s})^2} = 5.7 \text{ m/s}$$

$$\theta = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{5.1 \text{ m/s}}{2.5 \text{ m/s}} = 64^\circ.$$

Note that $\vec{v}$ is in the direction of motion (tangent to the curve that represents the path of the crate).

Can you find the location of the crate at $t = 3.0$ s?

4-3 PROJECTILE MOTION

A common example of motion in two dimensions is projectile motion near the Earth's surface, in which a projectile, such as a golf ball or a baseball, is launched in an arbitrary direction. For the time being we neglect air resistance (a nonconstant force), which simplifies the calculation.

Figure 4-4a shows the initial motion of the projectile at the instant of launch. Its initial velocity is $\vec{v}_0$ directed at an angle ϕ_0 from the horizontal. We choose our coordinate system with the x axis horizontal, its positive direction corresponding to the horizontal component of the initial velocity. The y axis is vertical, with its positive direction upward. We also place the origin of the coordinate system at the location of launch, so that $x_0 = 0$ and $y_0 = 0$. The components of the initial velocity are

$$v_{0x} = v_0 \cos \phi_0, \quad v_{0y} = v_0 \sin \phi_0. \quad (4-6)$$

The free-body diagram of the projectile (of mass m) is shown in Fig. 4-4b. Gravity is the only force that acts. It is

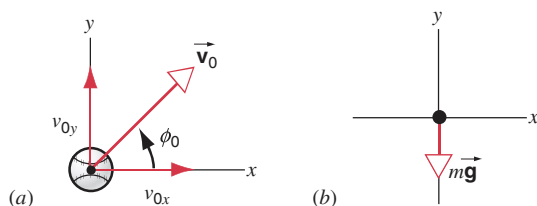


FIGURE 4-4. (a) A projectile is launched with initial velocity $\vec{v}_0$. (b) The free-body diagram of the projectile.

a constant force, having a downward direction and the same magnitude mg everywhere on the path of the projectile, no matter what its location or direction of motion. With our choice of coordinate system, the components of the net force are

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = -mg \quad (4-7)$$

and the vector components of Newton's second law (Eqs. 4-4) then give

$$a_x = \frac{\sum F_x}{m} = 0 \quad \text{and} \quad a_y = \frac{\sum F_y}{m} = -g. \quad (4-8)$$

The horizontal component of the acceleration is zero everywhere along the path, and the vertical component of the acceleration is $-g$ everywhere.

Applying the formulas for constant acceleration from Section 4-1, we obtain

velocity components:

$$(a) \quad v_x = v_{0x}, \quad (b) \quad v_y = v_{0y} - gt \quad (4-9)$$

position components:

$$(a) \quad x = v_{0x}t, \quad (b) \quad y = v_{0y}t - \frac{1}{2}gt^2 \quad (4-10)$$

Note that the horizontal component of the velocity remains constant (and equal to its initial value) throughout the flight. We know from Eqs. 4-4 that the x component of the acceleration can be affected *only* by a net force with an x component. In this case $\sum F_x = 0$, so v_x remains constant. The force in the y direction affects only y and v_y , not x or v_x .

The equations for the vertical motion (Eqs. 4-9b and 4-10b) are exactly those of free fall (Eqs. 2-29 and 2-30). In fact, if we observed the motion from a car traveling along the ground at velocity v_{0x} in the direction of the projectile, the motion would appear to be that of a projectile thrown vertically upward with initial speed v_{0y} .

Figure 4-5 shows the motion. At any point, the magnitude of the velocity vector is

$$v = \sqrt{v_x^2 + v_y^2} \quad (4-11)$$

and its direction is given by

$$\tan \phi = \frac{v_y}{v_x}, \quad (4-12)$$

where ϕ is the angle that the velocity vector makes with the horizontal. At every point of the motion, the velocity vector is tangent to the path of the projectile.

Equations 4-10 give us x and y as functions of the common parameter t . By combining and eliminating t from them, we obtain

$$y = (\tan \phi_0)x - \frac{g}{2(v_0 \cos \phi_0)^2} x^2, \quad (4-13)$$

which relates y to x and is the equation of the *trajectory* of the projectile. Since v_0 , ϕ_0 , and g are constants, this equation has the form

$$y = bx - cx^2,$$

the equation of a parabola. Hence the trajectory of a projectile is parabolic, as shown in Fig. 4-5.

The *horizontal range* R of the projectile, as shown in Fig. 4-5, is defined as the distance along the horizontal where the projectile returns to the level from which it was launched. We can find the range by putting $y = 0$ into Eq. 4-13. One solution immediately arises at $x = 0$; the other gives the range:

$$R = \frac{2v_0^2}{g} \sin \phi_0 \cos \phi_0 = \frac{v_0^2}{g} \sin 2\phi_0, \quad (4-14)$$

using the trigonometric identity $\sin 2\theta = 2 \sin \theta \cos \theta$. Note that, for a given initial speed, we get the maximum range for $\phi_0 = 45^\circ$, such that $\sin 2\phi_0 = 1$.

Figure 4-6 shows a strobe photo of the path of a projectile that is not severely affected by air resistance. The path certainly appears parabolic in its shape. Figure 4-7 compares the motions of a projectile dropped from rest to one simultaneously fired horizontally. Here you can see directly the predictions of Eqs. 4-10 when $\phi_0 = 0$. Note that (1) the horizontal motion of ball 2 does indeed follow Eq. 4-10: its x coordinate increases by equal amounts in equal intervals of

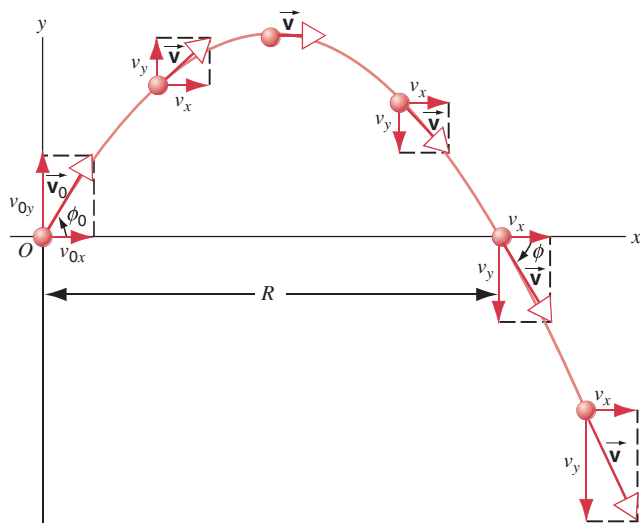


FIGURE 4-5. The trajectory of a projectile, showing the initial velocity $\vec{v}_0$ and its components and also the velocity $\vec{v}$ and its components at five later times. Note that $v_x = v_{0x}$ throughout the flight. The horizontal distance R is the range of the projectile.

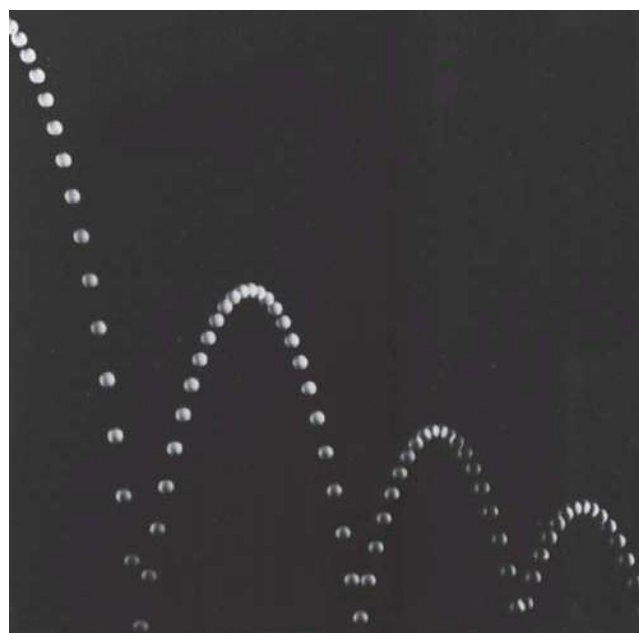


FIGURE 4-6. A strobe photo of a golf ball (which enters the photo from the left) bouncing off a hard surface. Between impacts, the ball shows the parabolic path characteristic of projectile motion. Why do you suppose the height of successive bounces is decreasing? (Chapter 6 may provide the answer.)

time, independent of the y motion; and (2) the y motions of the two projectiles are identical: the vertical increments of the position of the two projectiles are the same, independent of the horizontal motion of one of them.

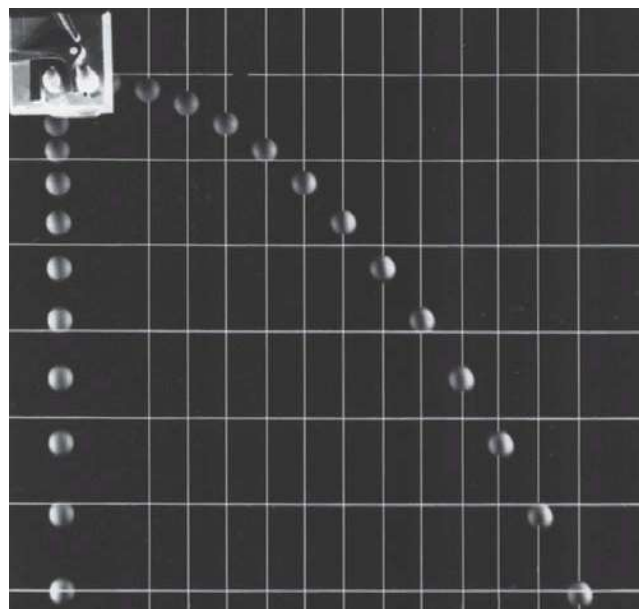


FIGURE 4-7. One ball is released from rest at the same instant that a second ball is fired to the right. Note that both balls fall at exactly the same rate; the horizontal motion of ball 2 does not affect its vertical rate of fall. The exposures in this strobe photo were taken at intervals of $1/30$ s. Does the horizontal velocity of the second ball appear to be constant?

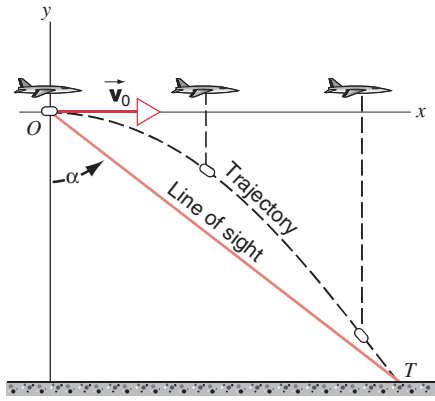


FIGURE 4-8. Sample Problem 4-3.

SAMPLE PROBLEM 4-3. In a contest to drop a package on a target, one contestant's plane is flying at a constant horizontal velocity of 155 km/h at an elevation of 225 m toward a point directly above the target. At what angle of sight α should the package be released to strike the target (Fig. 4-8)?

Solution We choose a reference frame fixed with respect to the Earth, its origin O being the release point. The motion of the package at the moment of release is the same as that of the plane. Hence the initial package velocity $\vec{v}_0$ is horizontal and its magnitude is 155 km/h. The angle of projection ϕ_0 is zero.

We find the time of fall from Eq. 4-10b. With $v_{0y} = 0$ and $y = -225$ m at the ground this gives

$$t = \sqrt{-\frac{2y}{g}} = \sqrt{-\frac{(2)(-225 \text{ m})}{9.8 \text{ m/s}^2}} = 6.78 \text{ s}.$$

Note that the time of fall does not depend on the speed of the plane for a horizontal projection.

The horizontal distance traveled by the package in this time is given by Eq. 4-10a:

$$\begin{aligned} x &= v_{0x}t = (155 \text{ km/h})(1 \text{ h}/3600 \text{ s})(6.78 \text{ s}) \\ &= 0.292 \text{ km} = 292 \text{ m} \end{aligned}$$

so that the angle of sight (Fig. 4-8) should be

$$\alpha = \tan^{-1} \frac{x}{|y|} = \tan^{-1} \frac{292 \text{ m}}{225 \text{ m}} = 52^\circ.$$

Does the motion of the package appear to be parabolic when viewed from a reference frame fixed with respect to the plane? (Can you recall having seen films of bombs dropping from a plane, taken by a camera either on that plane or on another plane flying a parallel course at the same speed?)

SAMPLE PROBLEM 4-4. A soccer player kicks a ball at an angle of 36° from the horizontal with an initial speed of 15.5 m/s. Assuming that the ball moves in a vertical plane, find (a) the time t_1 at which the ball reaches the highest point of its trajectory, (b) its maximum height, (c) its time of flight and range, and (d) its velocity when it strikes the ground.

Solution (a) The vertical component of the initial velocity is $v_{0y} = v_0 \sin \phi_0 = (15.5 \text{ m/s}) \sin 36^\circ = 9.1 \text{ m/s}$. At the top of its

trajectory, $v_y = 0$. Solving Eq. 4-9b for the time and substituting the numerical values we obtain:

$$t_1 = \frac{v_{0y} - v_y}{g} = \frac{9.1 \text{ m/s} - 0}{9.8 \text{ m/s}^2} = 0.93 \text{ s}.$$

(b) The maximum height $y_{\max}$ is reached at $t_1 = 0.93$ s. Using Eq. 4-10b, we have

$$\begin{aligned} y_{\max} &= v_{0y}t_1 - \frac{1}{2}gt_1^2 \\ &= (9.1 \text{ m/s})(0.93 \text{ s}) - \frac{1}{2}(9.8 \text{ m/s}^2)(0.93 \text{ s})^2 = 4.2 \text{ m}. \end{aligned}$$

(c) To find the total time of flight t_2 , we set $y = 0$ in Eq. 4-10b and, after eliminating the solution $t = 0$ (which reminds us that the ball did indeed start at $y = 0$ when $t = 0$) we solve to find the other time when the ball is at $y = 0$:

$$t_2 = \frac{2v_{0y}}{g} = \frac{2(9.1 \text{ m/s})}{9.8 \text{ m/s}^2} = 1.86 \text{ s}.$$

Note that $t_2 = 2t_1$, which must occur because the time required for the ball to go up (reach its maximum height from the ground) is the same as the time required for it to come down (reach the ground from its maximum height). The range is the horizontal distance traveled during the time t_2 :

$$x = v_{0x}t_2 = (v_0 \cos \phi_0)t_2 = (15.5 \text{ m/s})(\cos 36^\circ)(1.86 \text{ s}) = 23.3 \text{ m}.$$

(d) To find the velocity of the ball when it strikes the ground, we use Eq. 4-9a to obtain v_x , which remains constant throughout the flight:

$$v_x = v_{0x} = v_0 \cos \phi_0 = (15.5 \text{ m/s})(\cos 36^\circ) = 12.5 \text{ m/s},$$

and from Eq. 4-9b we obtain v_y for $t = t_2$,

$$\begin{aligned} v_y &= v_{0y} - gt = 9.1 \text{ m/s} - (9.8 \text{ m/s}^2)(1.86 \text{ s}) \\ &= -9.1 \text{ m/s}. \end{aligned}$$

Hence, the velocity has magnitude given by

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(12.5 \text{ m/s})^2 + (-9.1 \text{ m/s})^2} = 15.5 \text{ m/s},$$

and direction given by

$$\tan \phi = v_y/v_x = (-9.1 \text{ m/s})/(12.5 \text{ m/s}),$$

so that $\phi = -36^\circ$, or 36° clockwise from the x axis. Note that $\phi = -\phi_0$, as we expect from symmetry (Fig. 4-5).

The final speed turned out to be equal to the initial speed. Can you explain this? Is it a coincidence?

Shooting a Falling Target

In a favorite lecture demonstration an air gun is sighted at an elevated target, which is released in free fall by a trip mechanism as the "bullet" leaves the muzzle. No matter what the initial speed of the bullet, it always hits the falling target.

To understand this surprising outcome, consider that if there were no acceleration due to gravity, the target would not fall and the bullet would move along the line of sight directly into the target (Fig. 4-9). The effect of gravity is to cause each body to accelerate down at the same rate from the position it would otherwise have had. Therefore, in the time t , the bullet will fall a distance $\frac{1}{2}gt^2$ from the position it would have had

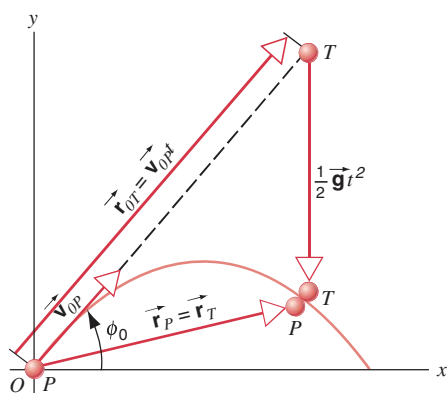


FIGURE 4-9. In the motion of a projectile, its displacement from the origin at any time t can be thought of as the sum of two vectors: $\vec{v}_{0P}t$, directed along $\vec{v}_{0P}$, and $\frac{1}{2}\vec{g}t^2$, directed down.

along the line of sight and the target will fall the same distance from its starting point. When the bullet reaches the line of fall of the target, it will be the same distance below the target's initial position as the target is and hence the collision. If the bullet moves faster than shown in the figure (v_0 larger), it will have a greater range and will cross the line of fall at a higher point; but since it gets there sooner, the target will fall a correspondingly smaller distance in the same time and collide with it. A similar argument holds for slower speeds.

For an equivalent analysis, let us use Eq. 4-2

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

to describe the positions of the projectile and the target at any time t . For the projectile P , $\vec{r}_0 = 0$ and $\vec{a} = \vec{g}$, and we have

$$\vec{r}_P = \vec{v}_{0P}t + \frac{1}{2}\vec{g}t^2. \quad (4-15)$$

For the target T , $\vec{r}_0 = \vec{r}_{0T}$, $\vec{v}_0 = 0$, and $\vec{a} = \vec{g}$, leading to

$$\vec{r}_T = \vec{r}_{0T} + \frac{1}{2}\vec{g}t^2. \quad (4-16)$$

If there is a collision, we must have $\vec{r}_P = \vec{r}_T$. Inspection shows that this will always occur at a time t given by $\vec{r}_{0T} = \vec{v}_{0P}t$ —that is, in the time $t (= r_{0T}/v_{0P})$ that it would take for an unaccelerated projectile to travel to the target position along the line of sight. Because multiplying a vector by a positive scalar gives another vector in the same direction, the equation $\vec{r}_{0T} = \vec{v}_{0P}t$ tells us that $\vec{r}_{0T}$ and $\vec{v}_{0P}$ must be in the same direction. That is, the gun must be aimed at the initial position of the target.

4-4 DRAG FORCES AND THE MOTION OF PROJECTILES (Optional)

Raindrops fall from clouds whose height h above the ground is about 2 km. Using our equations for freely falling bodies (Eqs. 2-29 and 2-30), we expect the raindrop to strike the ground with a speed of $v \approx 200$ m/s, or about 440 mi/h. The impact of a projectile, even a raindrop, at that speed

would be lethal; since raindrops move at much slower speeds, we have obviously made an error in the analysis.

The error occurs when we neglect the effect of the frictional force exerted by the air on the falling raindrop. This frictional force is an example of a *drag force*, experienced by any object that moves through a fluid medium, such as air or water. Drag forces have important effects on a variety of objects, such as baseballs, which deviate considerably from the ideal drag-free trajectory, and downhill skiers, who try to streamline their bodies and skiing position to reduce the drag. Drag forces must be taken into account in the design of aircraft and seacraft. From the standpoint of falling bodies, from raindrops to skydivers, drag forces prevent the velocity from increasing without limit and they impose a maximum or *terminal* speed that can be attained by a falling body.

One particular characteristic of drag forces is that they depend on the speed: the faster an object moves, the greater is the drag force. We therefore cannot use our formulas for constant acceleration to analyze motion under drag forces.

We illustrate the technique for handling problems with nonconstant forces by considering a body of mass m that is dropped from rest. We assume that the magnitude of the drag force D depends linearly on the speed:

$$D = bv \quad (4-17)$$

and always acts in a direction opposite to the motion. The constant b depends on properties of the falling object (its size and shape, for instance) and on the properties of the fluid (air, in this case) through which the object falls. Our goal is to find the velocity of the falling object as a function of the time.

Figure 4-10 shows the free-body diagram, which changes with time as the object falls. When the object is released, $D = 0$ (because $v_y = 0$), and D increases as the object falls. As D continues to increase, at some point it will equal the weight of the object, and at that point there is no net force acting on the object; its acceleration is zero, so its velocity remains constant, as does the drag force. From that time on, the object falls with constant velocity, which is the terminal velocity.

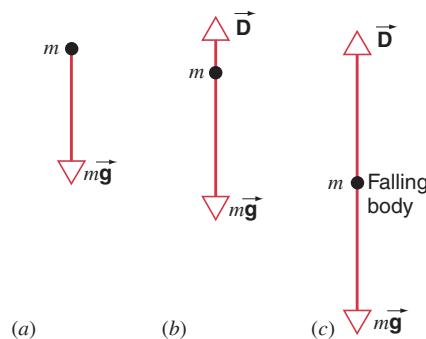


FIGURE 4-10. Forces acting on a body falling in air. (a) At the instant of release, $v_y = 0$ and there is no drag force. (b) The drag force increases as the body gains speed. (c) Eventually the drag force equals the weight; for all later times it remains equal to the weight and the body falls at its constant terminal speed.

We choose the y axis to be vertical and the positive direction to be downward. (The choice of direction is arbitrary, and here it is convenient to work with positive velocity and acceleration components.) With the weight mg acting downward and the drag force bv_y acting upward, the net force is then $\Sigma F_y = mg - bv_y$, so that Newton's second law $\Sigma F_y = ma_y$ gives

$$mg - bv_y = ma_y \quad (4-18)$$

or

$$a_y = g - \frac{bv_y}{m}. \quad (4-19)$$

Our goal is to find the velocity as a function of the time. We begin by substituting $a_y = dv_y/dt$ in Eq. 4-19, which gives

$$\frac{dv_y}{g - bv_y/m} = dt. \quad (4-20)$$

With $v_y = 0$ at time $t = 0$, we seek the velocity v_y at time t . We can therefore integrate the left side of Eq. 4-20 from velocity 0 to v_y and the right side from time 0 to t . (See Eq. 5 of Appendix I.) The result is

$$-\frac{m}{b} \ln \left(\frac{mg - bv_y}{mg} \right) = t \quad (4-21)$$

and solving for v_y we obtain

$$v_y(t) = \frac{mg}{b} (1 - e^{-bt/m}). \quad (4-22)$$

This is the expression for the velocity as a function of time.

It is interesting to examine this result in the two limiting cases of small and large values of t . The velocity starts with $v_y = 0$ at $t = 0$. Just after $t = 0$, near the beginning of the projectile's fall, we can find the velocity by approximating the exponential function using $e^{-x} \approx 1 - x$ for small x ($x \ll 1$). This gives

$$v_y(t) \approx \frac{mg}{b} \left[1 - \left(1 - \frac{bt}{m} \right) \right] = gt \quad (\text{small } t), \quad (4-23)$$

which agrees with Eq. 2-29 when $v_{0y} = 0$ (recalling that here we chose the positive y direction to be downward). Early in the motion, when the velocity is small and the drag force has not increased significantly, the object is approximately in free fall.

For large t , the exponential approaches zero ($e^{-x} \rightarrow 0$ as $x \rightarrow \infty$) and the magnitude of the velocity approaches the *terminal speed* given by

$$v_T = \frac{mg}{b}. \quad (4-24)$$

We can also find the terminal speed directly from Eq. 4-19—when the speed increases to the point at which the drag force and the weight are equal, $a_y = 0$ and Eq. 4-19 then gives Eq. 4-24.

We see that, just as we expect, the larger is the drag force coefficient b , the smaller is the terminal speed. The

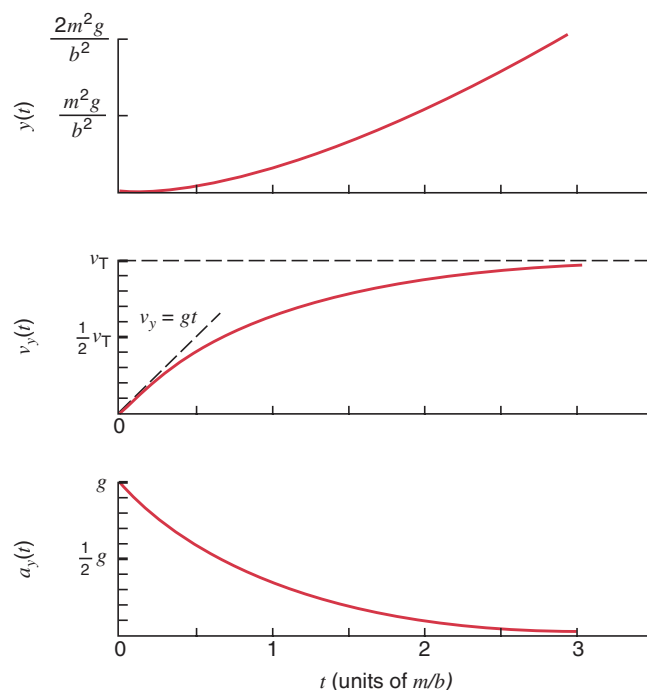


FIGURE 4-11. Position, velocity, and acceleration for a falling body subject to a drag force. Note that the acceleration starts at g and falls to zero; the velocity starts at zero and approaches v_T . Note also that $y(t)$ becomes nearly linear at large t , as we expect for motion with constant velocity.

terminal speed of a pebble falling in water is less than that of the same pebble falling in air, because the drag coefficient is much larger in water.

Now that we have an expression for $v_y(t)$, we can differentiate it to find $a_y(t)$ or integrate it to find $y(t)$. (See Problem 17.) Figure 4-11 shows the time dependence of y , v_y , and a_y .

A drag force proportional to v is representative of *viscous drag*, which is the force that might be experienced by a small particle falling through a thick fluid. Large objects

TABLE 4-1 Some Terminal Speeds in Air

Object	Terminal Speed (m/s)	95% Distance ^a (m)
16-lb shot	145	2500
Skydiver (typical)	60	430
Baseball	42	210
Tennis ball	31	115
Basketball	20	47
Ping-Pong ball	9	10
Raindrop (radius = 1.5 mm)	7	6
Parachutist (typical)	5	3

^aThis is the distance through which the body must fall from rest to reach 95% of its terminal speed.

Source: Adapted from Peter J. Brancazio, *Sport Science* (Simon & Schuster, 1984).

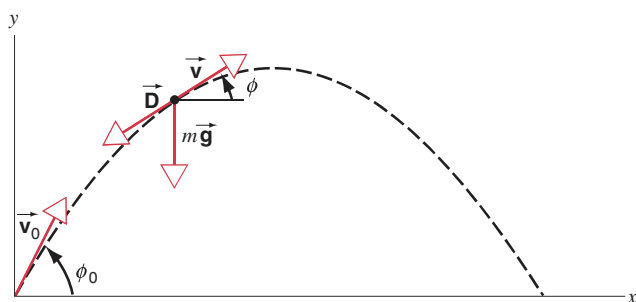


FIGURE 4-12. A projectile in motion. It is launched with velocity v_0 at an angle ϕ_0 with the horizontal. At a certain time later its velocity is $\vec{v}$ at the angle ϕ . The weight and the drag force (which always points in a direction opposite to $\vec{v}$) are shown at that time.

in air experience *aerodynamic drag*, in which D is proportional to v^2 . This case is more complicated mathematically, but it also yields a terminal speed (different from the terminal speed calculated for $D \propto v$).

Table 4-1 shows typical measured values of terminal speeds of different objects in air.

Projectile Motion with Air Resistance

Drag calculations are also important for two-dimensional projectile motion. A baseball, for example, leaves the bat with a speed of about 100 mi/h or 45 m/s. This is already greater than its terminal speed in air when dropped from rest (Table 4-1). The magnitude of the drag force $D = bv$ can be estimated from our previous calculation. From Eq. 4-24 we see that the constant b is the weight mg of the baseball (about 1.4 N, corresponding to a mass of 0.14 kg) divided by its terminal speed, 42 m/s. Thus $b = 0.033 \text{ N/(m/s)}$. If the ball travels at 45 m/s, it experiences a drag force bv with a magnitude of about 1.5 N, which is greater than its weight and therefore has a substantial effect on its motion.

Figure 4-12 shows the free-body diagram at a particular point in the baseball's trajectory. Like all frictional forces, $\vec{D}$ is in a direction opposite to $\vec{v}$, and we assume no wind is blowing. If we take $\vec{D} = -b\vec{v}$, we can use Newton's laws

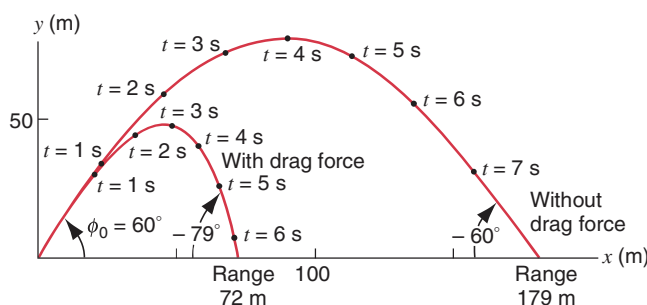


FIGURE 4-13. Projectile motion with and without a drag force, calculated for $v_0 = 45 \text{ m/s}$ and $\phi_0 = 60^\circ$.

to find an analytic solution for the trajectory, an example of which is illustrated in Fig. 4-13. When air resistance is taken into account, the range is reduced from 179 m to 72 m and the maximum height from 78 m to 48 m. Note also that the trajectory is no longer symmetric about the maximum; the descending motion is much steeper than the ascending motion. For $\phi_0 = 60^\circ$, the projectile strikes the ground at an angle of -79° , while in the absence of drag it would strike the ground at an angle equal to $-\phi_0 = -60^\circ$.

For other (and more realistic) choices for the drag force $\vec{D}$, the calculation must be done numerically.*

4-5 UNIFORM CIRCULAR MOTION

In projectile motion in the absence of air resistance, the acceleration is constant in both magnitude and direction, but the velocity changes in both magnitude and direction. We now examine a different case of motion in two dimensions in which a particle moves at constant *speed* in a circular path. As we shall see, both the velocity and acceleration are constant in magnitude, but both change their directions continuously. This situation is called *uniform circular motion*. Examples of this kind of motion may include Earth satellites and points on spinning rotors such as fans or merry-go-rounds. In fact, to the extent that we can regard ourselves as particles, we are in uniform circular motion because of the rotation of the Earth.

As an example, imagine you are swinging a ball on a string in a horizontal plane, as in Fig. 4-14. (We neglect the drag force and the force of gravity for the time being.) As you swing the ball, your fingers are exerting a force on the string (and the string in turn exerts a force on the ball). If you were to loosen your grip on the string slightly, the string would slide between your fingers and the ball would move away from the center of the circle, so to prevent this

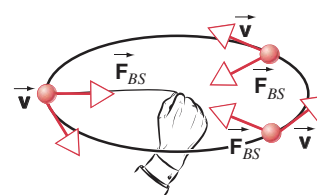


FIGURE 4-14. A ball on a string is whirled in a horizontal circle. Vectors representing the velocity and the force of the string on the ball are shown at three different instants.

* You can find more information about this calculation in "Trajectory of a Fly Ball," by Peter J. Brancazio, *The Physics Teacher*, January 1985, p. 20. For an interesting collection of articles about similar problems, see *The Physics of Sports*, edited by Angelo Armenti, Jr. (American Institute of Physics, 1992). See <http://www.physics.uoguelph.ca/fun/JAVA/trajplot/trajplot.html> for an interesting program that allows you to display the trajectories of a projectile for various choices of launch angle and air resistance.

from happening your fingers must be exerting an inward force on the string.

A similar example occurs in planetary motion. As the Moon moves in its orbit about the Earth, the Earth exerts a gravitational force that always points toward the center of the Earth (Fig. 4-15).

In both of these cases, the force is constant in magnitude but varies in direction as the object revolves in its circular path. Because the force always points toward the center of the circle, it is known as a *centripetal* (“seeking the center”) force. Since no other force acts, the acceleration must also point toward the center of the circle (centripetal acceleration). From the geometry of the circular motion, we can obtain an expression for the centripetal acceleration.

It is important to note that, in uniform circular motion, the magnitude of the velocity stays constant but the particle is still accelerating because the *direction* of its velocity is changing. Even though we usually associate an acceleration with a change in the magnitude of $\vec{v}$, there must also be an acceleration present to change the direction of $\vec{v}$.

To find the relationship between this acceleration and the constant magnitude of the velocity, consider the geometry of Fig. 4-16. A particle is moving in a circle of radius r . We set up an xy coordinate system with its origin at the center of the circle, and we examine the motion of the particle at two locations: at P_1 , where its velocity is $\vec{v}_1$, and at P_2 , where its velocity is $\vec{v}_2$. The points P_1 and P_2 are located symmetrically with respect to the y axis, with the radius to each location making an angle θ with the y axis.

The magnitudes of $\vec{v}_1$ and $\vec{v}_2$ are equal, but they have different directions, each being tangent to the circle at the location of the particle. The velocity components are:

$$\begin{aligned} v_{1x} &= +v \cos \theta & v_{1y} &= +v \sin \theta \\ v_{2x} &= +v \cos \theta & v_{2y} &= -v \sin \theta \end{aligned} \quad (4-25)$$

where we have used v to represent the common magnitude of $\vec{v}_1$ and $\vec{v}_2$.

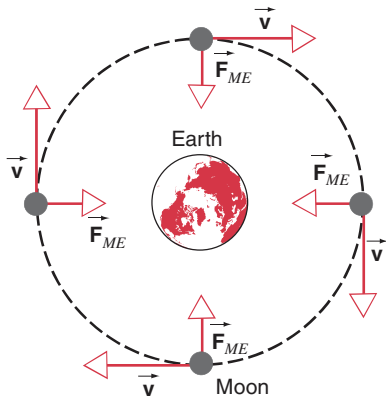


FIGURE 4-15. The Moon moves in its orbit around the Earth. The velocity and force vectors are shown at four different instants. The velocity is always tangent to the circular path, and the force on the Moon due to the Earth always points toward the center of the circle.

As the particle moves along the arc from P_1 to P_2 , it covers a distance of $2r\theta$ (where θ is measured in radians), and if it does so in a time interval Δt then its speed v is $2r\theta/\Delta t$. We can therefore express the time interval as

$$\Delta t = \frac{2r\theta}{v}. \quad (4-26)$$

Now we can find the components of the average acceleration. We use the definition of average acceleration from Eq. 2-14, $\vec{a}_{av} = \Delta\vec{v}/\Delta t$ where $\Delta\vec{v}$ means $\vec{v}_2 - \vec{v}_1$. The x component of the average acceleration is then

$$a_{av,x} = \frac{v_{2x} - v_{1x}}{\Delta t} = \frac{v \cos \theta - v \cos \theta}{\Delta t} = 0. \quad (4-27)$$

As shown in Fig. 4-16, the x components of the velocity are the same at P_1 and P_2 , so it is not surprising that the x component of the average acceleration is zero in that interval. The y component of the average acceleration is

$$\begin{aligned} a_{av,y} &= \frac{v_{2y} - v_{1y}}{\Delta t} = \frac{-v \sin \theta - v \sin \theta}{\Delta t} \\ &= \frac{-2v \sin \theta}{2r\theta/v} = -\left(\frac{v^2}{r}\right)\left(\frac{\sin \theta}{\theta}\right). \end{aligned} \quad (4-28)$$

We can find the instantaneous acceleration from this result by taking the limit as the time interval approaches zero. Equivalently, we can let the angle θ go to zero, so that P_1 and P_2 both approach P , which gives

$$a_y = \lim_{\theta \rightarrow 0} \left[-\left(\frac{v^2}{r}\right)\left(\frac{\sin \theta}{\theta}\right) \right] = -\left(\frac{v^2}{r}\right) \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right).$$

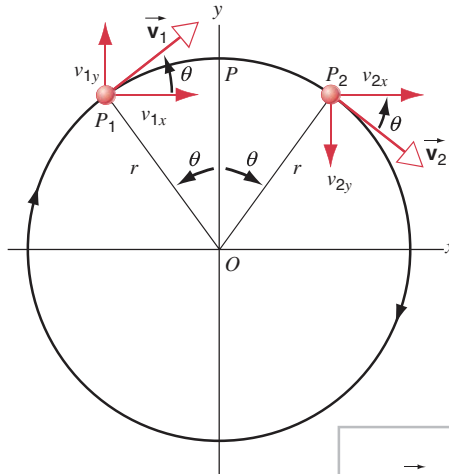


FIGURE 4-16. A particle moves at constant speed in a circle of radius r . It is shown at locations P_1 and P_2 , where the radius makes equal angles θ on opposite sides of the y axis. The inset shows the vector $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$; this vector always points toward the center of the circle, no matter where we choose points P_1 and P_2 .

For small angles, $\sin \theta \approx \theta$ (in radians), so the limit approaches the value 1. The y component of the instantaneous acceleration at P is then $a_y = -v^2/r$, the minus sign indicating that the acceleration at P points in the negative y direction—that is, toward the center of the circle.

Point P is an arbitrary point on the circle. We could have repeated the above calculation for any point on the circle and we would have obtained the same result: *the acceleration points toward the center of the circle and has magnitude v^2/r* . This is a general result for any particle that moves in a circle at constant speed; the centripetal acceleration is

$$a_c = \frac{v^2}{r}. \quad (4-29)$$

The centripetal acceleration is sometimes also called the *radial* acceleration, since its direction is always along a radius of the circle. In Fig. 4-16 you can see that the direction of $\vec{a}$ is the same as the direction of $\Delta\vec{v}$, just as the vector relationship of Eq. 2-14 requires.

Both in free fall and in projectile motion $\vec{a}$ is constant in direction and magnitude, and we can use the equations developed for constant acceleration. We cannot use these equations for uniform circular motion because $\vec{a}$ varies in direction and is therefore not constant.

The units of centripetal acceleration are the same as those of an acceleration resulting from a change in the magnitude of a velocity. Dimensionally, we have

$$[a] = \frac{[v^2]}{[r]} = \frac{(L/T)^2}{L} = \frac{L}{T^2},$$

which are the usual dimensions of acceleration. The units therefore may be m/s^2 , km/h^2 , or similar units of dimension L/T^2 .

The acceleration resulting from a change in direction of a velocity is just as real and just as much an acceleration in every sense as that arising from a change in magnitude of a velocity. By definition, acceleration is the time rate of change of velocity, and velocity, being a vector, can change in direction as well as magnitude. If a physical quantity is a vector, its directional aspects cannot be ignored, for their effects will prove to be every bit as important and real as those produced by changes in magnitude.

According to Newton's second law in its vector form ($\Sigma \vec{F} = m\vec{a}$), the acceleration and the net force must have the same direction. In the case of circular motion at constant speed, the net force must thus point toward the center of the circle. For now we will write this result in terms of magnitudes: $|\Sigma \vec{F}| = ma$. For uniform circular motion, $a = a_c = v^2/r$ and so

$$|\Sigma \vec{F}| = \frac{mv^2}{r}. \quad (4-30)$$

The quantity on the left side of Eq. 4-30 is sometimes called the “centripetal force.” The centripetal force is not a new kind of force. When a particle moves in a circular path

at constant speed, several forces may act on it. The resultant of all those forces must point toward the center of the circle, and we call that resultant the centripetal force. Newton's second law then gives the magnitude and direction of the acceleration.

In Fig. 4-14, the string provides the centripetal force that acts on the ball; in Fig. 4-15, the gravitational force of the Earth provides the centripetal force that acts on the Moon. To label a force as “centripetal” simply means that it acts toward the center of the circle, but that label tells us nothing about the nature of the force or the body that is exerting it. All forces, including those that act centripetally, must always be associated with a specific body in the environment. The centripetal force can be any type of force and might, for example, be provided by the action of gravity, strings, springs, or electric charges. As indicated in Eq. 4-30, it can also be a combination of two or more forces, as long as the direction of the resultant force is toward the center of the circle.

In this section we have discussed uniform circular motion as an example of a case in which vector laws are essential to understand motion in two dimensions. More general vector techniques can be used to describe the case in which the acceleration has both radial and tangential components. These techniques are described in Chapter 8.

SAMPLE PROBLEM 4-5. The Moon revolves about the Earth, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius $r = 238,000$ miles. What is the magnitude of the gravitational force exerted on the Moon by the Earth?

Solution We have $r = 238,000 \text{ mi} = 3.82 \times 10^8 \text{ m}$. From Appendix C, we find the mass of the Moon is $m = 7.36 \times 10^{22} \text{ kg}$. The time for one complete revolution, called the period, is $T = 27.3 \text{ d} = 2.36 \times 10^6 \text{ s}$. The speed of the Moon (assumed constant) is therefore

$$v = \frac{2\pi r}{T} = \frac{2\pi(3.82 \times 10^8 \text{ m})}{2.36 \times 10^6 \text{ s}} = 1018 \text{ m/s}.$$

The centripetal force is provided by the gravitational force on the Moon by the Earth:

$$\begin{aligned} F_{ME} &= \frac{mv^2}{r} = \frac{(7.36 \times 10^{22} \text{ kg})(1018 \text{ m/s})^2}{3.82 \times 10^8 \text{ m}} \\ &= 2.00 \times 10^{20} \text{ N}. \end{aligned}$$

SAMPLE PROBLEM 4-6. A satellite of mass 1250 kg is to be placed in a circular orbit at a height $h = 210 \text{ km}$ above the Earth's surface, where $g = 9.2 \text{ m/s}^2$. (a) What is the weight of the satellite at this altitude? (b) With what tangential speed must it be inserted into its orbit? The Earth's radius is $R = 6370 \text{ km}$.

Solution (a) The weight of the satellite is

$$W = mg = (1250 \text{ kg})(9.2 \text{ m/s}^2) = 1.15 \times 10^4 \text{ N}.$$

(b) The weight is the force of gravity F_{SE} exerted on the satellite by the Earth. Since this is the only force that acts on the satellite,

it must provide the centripetal force. Solving Eq. 4-30 for the tangential speed v , we obtain (with $r = R + h$):

$$v = \sqrt{\frac{F_{SEr}}{m}} = \sqrt{\frac{(1.15 \times 10^4 \text{ N})(6370 \text{ km} + 210 \text{ km})}{1250 \text{ kg}}} \\ = 7780 \text{ m/s} = 17,400 \text{ mi/h.}$$

At this speed, the satellite completes one orbit every 1.48 h.

4-6 RELATIVE MOTION

In Section 3-2 we discussed inertial frames and how observers in motion relative to one another will deduce identical accelerations if they are both in inertial frames. These observers will thus agree on the application of Newton's second law.

In this section, we expand the comparison of observations from different inertial frames using vector considerations. As before, we consider the description of the motion of a single particle by two observers who are in uniform motion relative to one another. The two observers might be, for example, a person in a car moving at constant velocity along a long, straight road and another person standing at rest on the ground. The particle they are both observing might be a ball tossed in the air or another moving car.

We call the two observers S and S' . Each has a corresponding reference frame to which is attached a Cartesian coordinate system. For convenience, we assume the observers to be located at the origins of their respective coordinate systems. We make only one restriction on this situation: *the relative velocity between S and S' must be a constant*. Here we mean constant in both magnitude and direction. Note that this restriction does not include the motion of the particle being observed by S and S' . The particle need not necessarily be moving with constant velocity, and indeed the particle may well be accelerating.

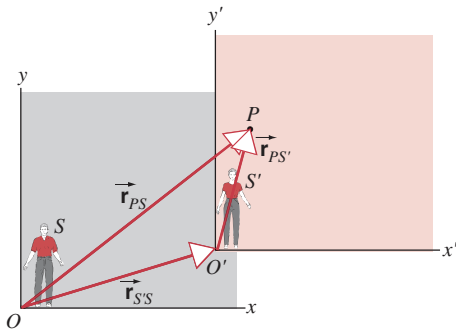


FIGURE 4-17. Observers S and S' , who are moving with respect to each other, observe the same moving particle P . At the time shown, they measure the position of the particle with respect to the origins of their coordinate systems to be $\vec{r}_{PS}$ and $\vec{r}_{PS'}$, respectively. At this same instant, observer S measures the position of S' with respect to the origin O to be $\vec{r}_{S'S}$.

Figure 4-17 shows, at a particular time t , the two coordinate systems belonging to S and S' . For simplicity, we consider motion in only two dimensions, the common xy and $x'y'$ planes shown in Fig. 4-17. The origin of the S' system is located with respect to the origin of the S system by the vector $\vec{r}_{S'S}$. Note in particular the order of the subscripts we use to label the vector: the first subscript gives the system being located (in this case, the coordinate system of S') and the second subscript gives the system with respect to which we are doing the locating (in this case, the coordinate system of S). The vector $\vec{r}_{S'S}$ would then be read as “the position of S' with respect to S .”

Figure 4-17 also shows a particle P in the common xy and $x'y'$ planes. Both S and S' locate the particle P with respect to their coordinate systems. According to S , the particle P is at the position indicated by the vector $\vec{r}_{PS}$, whereas according to S' the particle P is at $\vec{r}_{PS'}$. From Fig. 4-17 we can deduce the following relationship among the three vectors:

$$\vec{r}_{PS} = \vec{r}_{S'S} + \vec{r}_{PS'} = \vec{r}_{PS'} + \vec{r}_{S'S}, \quad (4-31)$$

where we have used the commutative law of vector addition to exchange the order of the two vectors. Once again, pay careful attention to the order of the subscripts. In words, Eq. 4-31 tells us: “the position of P as measured by S is equal to the position of P as measured by S' plus the position of S' as measured by S .”

Suppose the particle P is moving with velocity $\vec{v}_{PS'}$ according to S' . What velocity will S measure for the particle? To answer this question, we need only take the derivative with respect to time of Eq. 4-31, which gives

$$\frac{d\vec{r}_{PS}}{dt} = \frac{d\vec{r}_{PS'}}{dt} + \frac{d\vec{r}_{S'S}}{dt}.$$

The rate of change of each position vector gives the corresponding velocity, so that

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}. \quad (4-32)$$

Thus, at any instant, the velocity of P as measured by S is equal to the velocity of P as measured by S' plus the relative velocity of S' with respect to S . Although we have illustrated Eqs. 4-31 and 4-32 for motion in two dimensions, they hold equally well in three dimensions.

Equation 4-32 is a law of the *transformation of velocities*. It permits us to transform a measurement of velocity made by an observer in one frame of reference—say, S' —to another frame of reference—say, S —as long as we know the relative velocity between the two reference frames. It is a law firmly grounded both in the common sense of everyday experience and in the concepts of space and time that are essential to the classical physics of Galileo and Newton. In fact, Eq. 4-32 is often called the *Galilean form of the law of transformation of velocities*.

We consider here only the very important special case in which the two reference frames are moving at constant velocity with respect to one another. That is, $\vec{v}_{S'S}$ is constant

both in magnitude and direction. The velocities $\vec{v}_{PS}$ and $\vec{v}_{PS'}$ that S and S' measure for the particle P may not be constant, and of course they will in general not be equal to one another. If, however, one of the observers—say, S' —measures a velocity that is constant in time, then both terms on the right-hand side of Eq. 4-32 are independent of time and therefore the left side of Eq. 4-32 must also be independent of time. Thus, if one observer concludes that the particle moves with constant velocity, then all other observers conclude the same, as long as the other observers are in frames of reference that move at constant velocity with respect to the frame of the first observer.

We can see this in a more formal way by differentiating Eq. 4-32:

$$\frac{d\vec{v}_{PS}}{dt} = \frac{d\vec{v}_{PS'}}{dt} + \frac{d\vec{v}_{S'S}}{dt}. \quad (4-33)$$

The last term of Eq. 4-33 vanishes, because we assume that the relative velocity of the two reference frames is a constant. Thus

$$\frac{d\vec{v}_{PS}}{dt} = \frac{d\vec{v}_{PS'}}{dt}.$$

Replacing these two derivatives of velocity with the corresponding accelerations, we obtain

$$\vec{a}_{PS} = \vec{a}_{PS'}. \quad (4-34)$$

The accelerations of P measured by the two observers are identical!

Equation 4-34 indicates directly why Newton's laws can be equally well applied by observers in any inertial frame. If our observers deduce identical accelerations for the moving particle, then they will agree on the results of applying $\vec{F} = m\vec{a}$. If observer S successfully tests to determine that Newton's laws are valid, then all other observers whose reference frames move relative to S with a velocity that is constant in both magnitude and direction will likewise find Newton's laws to be valid.

SAMPLE PROBLEM 4-7. The compass in an airplane indicates that it is headed due east; its air speed indicator reads 215 km/h. A steady wind of 65 km/h is blowing due north. (a) What is the velocity of the plane with respect to the ground? (b) If the pilot wishes to fly due east, what must be the heading? That is, what must the compass read?

Solution (a) The moving “particle” in this problem is the plane P . There are two reference frames, the ground (G) and the air (A). We let the ground be our S system and the air be the S' system, and by a simple change of notation, we can rewrite Eq. 4-32 as

$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}.$$

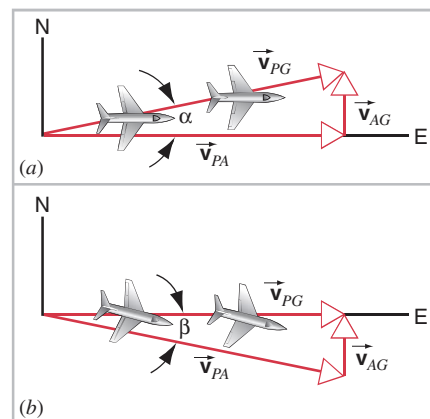


FIGURE 4-18. Sample Problem 4-7. (a) A plane, heading due east, is blown to the north. (b) To travel due east, the plane must head into the wind.

Figure 4-18a shows these vectors, which form a right triangle. The terms are, in sequence, the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground (that is, the wind velocity). Note the orientation of the plane, which is consistent with a due east reading on its compass.

The magnitude of the ground velocity (the ground speed) is found from

$$v_{PG} = \sqrt{v_{PA}^2 + v_{AG}^2} = \sqrt{(215 \text{ km/h})^2 + (65 \text{ km/h})^2} = 225 \text{ km/h}.$$

The angle α in Fig. 4-18a follows from

$$\alpha = \tan^{-1} \frac{v_{AG}}{v_{PA}} = \tan^{-1} \frac{65 \text{ km/h}}{215 \text{ km/h}} = 16.8^\circ.$$

Thus, with respect to the ground, the plane is flying at 225 km/h in a direction 16.8° north of east. Note that its ground speed is greater than its air speed.

(b) In this case the pilot must head into the wind so that the velocity of the plane with respect to the ground points east. The wind remains unchanged and the vector diagram representing Eq. 4-32 is as shown in Fig. 4-18b. Note that the three vectors still form a right triangle, as they did in Fig. 4-18a, but in this case the hypotenuse is v_{PA} rather than v_{PG} .

The pilot's ground speed is now

$$v_{PG} = \sqrt{v_{PA}^2 - v_{AG}^2} = \sqrt{(215 \text{ km/h})^2 - (65 \text{ km/h})^2} = 205 \text{ km/h}.$$

As the orientation of the plane in Fig. 4-18b indicates, the pilot must head into the wind by an angle β given by

$$\beta = \sin^{-1} \frac{v_{AG}}{v_{PA}} = \sin^{-1} \frac{65 \text{ km/h}}{215 \text{ km/h}} = 17.6^\circ.$$

Note that, by heading into the wind as the pilot has done, the ground speed is now less than the air speed.

MULTIPLE CHOICE

4-1 Motion in Three Dimensions with Constant Acceleration

- An object moves in the xy plane with an acceleration that has a positive x component. At time $t = 0$ the object has a velocity given by $\vec{v} = 3\hat{i} + 0\hat{j}$.
 - What can be concluded about the y component of the acceleration?
 - The y component must be positive and constant.
 - The y component must be negative and constant.
 - The y component must be zero.
 - Nothing at all can be concluded about the y component.
 - What can be concluded about the y component of the velocity?
 - The y component must be increasing.
 - The y component must be constant.
 - The y component must be decreasing.
 - Nothing at all can be concluded about the variation of the y component.
 - What can be concluded about magnitude of the velocity?
 - The magnitude of the velocity must be increasing.
 - The magnitude of the velocity must be constant.
 - The magnitude of the velocity component must be decreasing.
 - Nothing at all can be concluded about the magnitude of the velocity.
- An object moves with a constant acceleration $\vec{a}$. Which of the following expressions are also constant?
 - $d|\vec{v}|/dt$
 - $|d\vec{v}/dt|$
 - $d(v^2)/dt$
 - $d(\vec{v}/|\vec{v}|)/dt$

4-2 Newton's Laws in Three-Dimensional Vector Form

- Suppose the net force $\vec{F}$ on an object is a nonzero constant. Which of the following could also be constant?
 - Position.
 - Speed.
 - Velocity.
 - Acceleration.
- Two forces of magnitude F_1 and F_2 are acting on an object. The magnitude of the net force F_{net} on the object will be in the range
 - $F_1 \leq F_{\text{net}} \leq F_2$.
 - $(F_1 - F_2)/2 \leq F_{\text{net}} \leq (F_1 + F_2)/2$.
 - $|F_1 - F_2| \leq F_{\text{net}} \leq |F_1 + F_2|$.
 - $F_1^2 - F_2^2 \leq (F_{\text{net}})^2 \leq F_1^2 + F_2^2$.

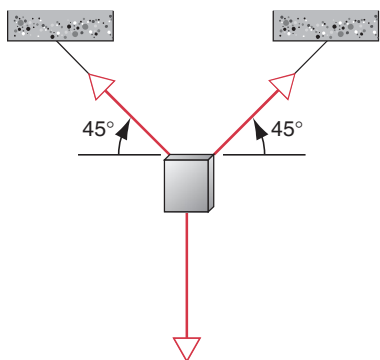


FIGURE 4-19. Multiple-choice questions 5 and 6.

- A small 2.0-kg object is suspended at rest from two strings as shown in Fig. 4-19. The magnitude of the force exerted by each string on the object is 13.9 N; the magnitude of the force of gravity is 19.6 N. The magnitude of the net force on the object is
 - 47.4 N.
 - 33.5 N.
 - 13.9 N.
 - 8.2 N.
 - 0 N.
- The string on the left in Fig. 4-19 suddenly snaps. At the instant when the string snaps, the magnitude of the net force on the object is
 - 47.4 N.
 - 33.5 N.
 - 13.9 N.
 - 8.2 N.
 - 0 N.

4-3 Projectile Motion

- A projectile is launched with initial velocity $\vec{v}_0$ at an angle ϕ_0 with the horizontal. Neglect air resistance.
 - Where in the motion is the net force on the projectile equal to zero?
 - Where in the motion is the acceleration of the projectile equal to zero?
 - Somewhere before reaching its maximum height.
 - At its highest point.
 - Somewhere after reaching its maximum height.
 - Nowhere in the trajectory.
- An object is launched into the air with an initial velocity given by $\vec{v}_0 = (4.9\hat{i} + 9.8\hat{j})$ m/s. Ignore air resistance.
 - At the highest point the magnitude of the velocity is
 - 0.
 - $\sqrt{4.9^2}$ m/s.
 - $\sqrt{9.8^2}$ m/s.
 - $\sqrt{4.9^2 + 9.8^2}$ m/s.
 - At $t = 0.5$ s the magnitude of the velocity is
 - $\sqrt{(4.9 + 9.8/2)^2}$ m/s.
 - $\sqrt{4.9^2 + (9.8/2)^2}$ m/s.
 - $\sqrt{(4.9/2)^2 + 9.8^2}$ m/s.
 - $\sqrt{(4.9/2)^2 + (9.8/2)^2}$ m/s.
- During the Battle of Tarawa in World War II, battleships fired ballistic projectiles at the Japanese garrisons on Betio from as far as 40 miles out at sea. Assuming no air resistance, and assuming the trajectories were chosen to give the optimum range,
 - the projectiles would have risen to a maximum altitude in the range of
 - 0 to $\frac{1}{2}$ mi.
 - $\frac{1}{2}$ to 2 mi.
 - 2 to 5 mi.
 - 5 to 8 mi.
 - 8 to 12 mi.
 - before returning to the ground.
 - What would be the approximate muzzle velocity of the projectiles?
 - 25,000 ft/s.
 - 2,500 ft/s.
 - 250 ft/s.
 - 25 ft/s.
- A projectile fired vertically up from a cannon rises 200 meters before returning to the ground. If the same cannon were to fire the same projectile at an angle, then the maximum range would be approximately
 - 200 m.
 - 400 m.
 - 800 m.
 - 1600 m.
 (Assume that air resistance is negligible.)

4-4 Drag Forces and the Motion of Projectiles

- Which graph in Fig. 4-20 best shows the velocity–time graph for an object launched vertically into the air when air resistance is given by $D = bv$? The dashed line shows the velocity graph if there were no air resistance.

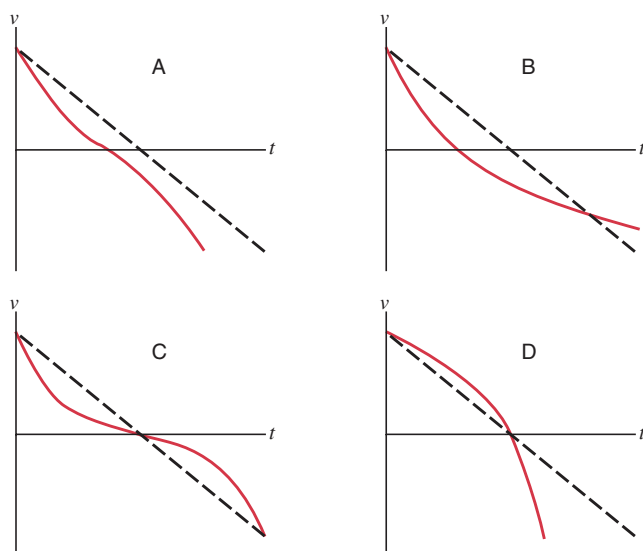


FIGURE 4-20. Multiple-choice question 11.

12. You calculate that to throw an object vertically to a height h it needs to be launched with an initial upward velocity v_0 , assuming no air resistance. The dashed lines in Fig. 4-21 show the motion according to this calculation. Which of the velocity–time graphs shows the motion of an object tossed with initial upward velocity v'_0 that will also rise to height h , but this time with air resistance?

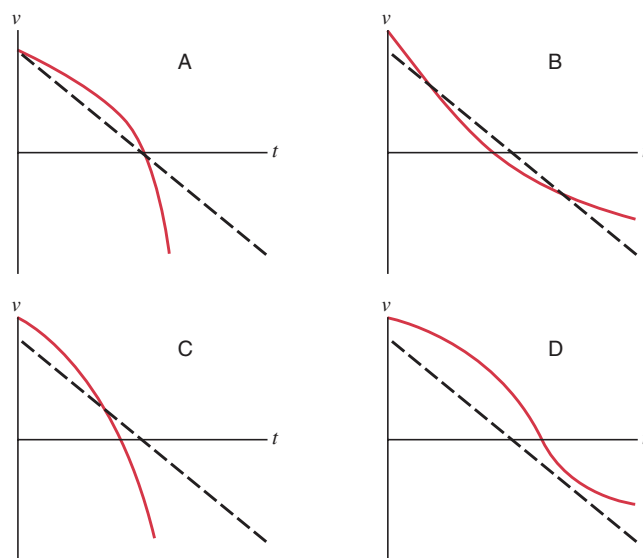


FIGURE 4-21. Multiple-choice question 12.

13. A parachutist jumps out of a plane. He falls freely for some time, and then opens his chute. Shortly after his chute inflates the parachutist
- keeps falling but quickly slows down.
 - momentarily stops, then starts falling again, but more slowly.
 - suddenly shoots upward, and then starts falling again, but more slowly.
 - suddenly shoots upward, and then starts falling again,

eventually acquiring the same speed as before the chute opened.

4-5 Uniform Circular Motion

14. Which statement is most correct?
- Uniform circular motion causes a constant force toward the center.
 - Uniform circular motion is caused by a constant force toward the center.
 - Uniform circular motion is caused by a constant magnitude *net* force toward the center.
 - Uniform circular motion is caused by a constant magnitude *net* force away from the center.
15. A puck is moving in a circle of radius r_0 with a constant speed v_0 on a level frictionless table. A string is attached to the puck, which holds it in the circle; the string passes through a frictionless hole and is attached on the other end to a hanging object of mass M . (See Fig. 4-22.)
- The puck is now made to move with a speed $v' = 2v_0$, but still in a circle. The mass of the hanging object is left unchanged. The acceleration a' of the puck and the radius r' of the circle are now given by
 - $a' = 4a_0$ and $r' = r_0$.
 - $a' = 2a_0$ and $r' = r_0$.
 - $a' = 2a_0$ and $r' = 2r_0$.
 - $a' = a_0$ and $r' = 4r_0$.
 - The puck continues to move at speed $v' = 2v_0$ in a circle, but now the mass of the hanging object is doubled. The acceleration a' of the puck and the radius r' of the circle are now given by
 - $a' = 4a_0$ and $r' = r_0$.
 - $a' = 2a_0$ and $r' = r_0$.
 - $a' = 2a_0$ and $r' = 2r_0$.
 - $a' = a_0$ and $r' = 4r_0$.

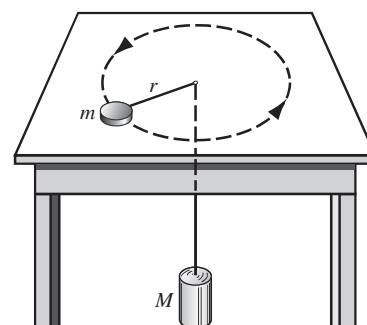


FIGURE 4-22. Multiple-choice question 15.

4-6 Relative Motion

16. An object has velocity $\vec{v}_1$ relative to the ground. An observer moving past with a constant velocity $\vec{v}_0$ relative to the ground measures the velocity of the object to be $\vec{v}_2$ (relative to the observer). The magnitudes of these velocities are related by
- $v_0 \leq v_1 + v_2$.
 - $v_1 \leq v_2 + v_0$.
 - $v_2 \leq v_0 + v_1$.
 - All the above are true.
17. (a) A boy sitting in a railroad car moving at constant velocity throws a ball straight up into the air, according to the person sitting next to him. Where will the ball fall?
- Behind him
 - In front of him
 - Into his hands
 - Beside him
- (b) Where would the ball fall if the train accelerates forward while the ball is in the air? If it rounds a curve?
- Behind him
 - In front of him
 - Into his hands
 - Beside him

QUESTIONS

1. A particle moves in three-dimensional space with a constant acceleration. Can the z component of the acceleration affect the x component of its location? Can the z component of the acceleration affect the y component of the velocity?
2. Describe a physical situation in which an object that moves in the xy plane might have an acceleration with a constant positive x component and a constant negative y component.
3. Can the acceleration of a body change its direction without its velocity changing direction?
4. Let $\vec{v}$ and $\vec{a}$ represent the velocity and acceleration, respectively, of an automobile. Describe circumstances in which (a) $\vec{v}$ and $\vec{a}$ are parallel; (b) $\vec{v}$ and $\vec{a}$ are antiparallel; (c) $\vec{v}$ and $\vec{a}$ are perpendicular to one another; (d) $\vec{v}$ is zero but $\vec{a}$ is not zero; (e) $\vec{a}$ is zero but $\vec{v}$ is not zero.
5. In Fig. 4-23, we show four forces that are equal in magnitude. What combination of three of these, acting together on the same particle, might keep that particle at rest?

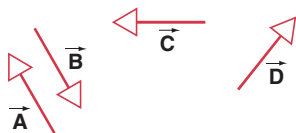


FIGURE 4-23. Question 5.

6. You shoot an arrow into the air and keep your eye on it as it follows a parabolic flight path to the ground. You note that the arrow turns in flight so that it is always tangent to its flight path. What makes it do that?
7. In a tug-of-war, three men pull on a rope to the left at A and three men pull to the right at B with forces of equal magnitude. Now a 5-lb weight is hung vertically from the center of the rope. (a) Can the men get the rope AB to be horizontal? (b) If not, explain. If so, determine the magnitude of the forces required at A and B to do this.
8. A tube in the shape of a rectangle with rounded corners is placed in a vertical plane, as shown in Fig. 4-24. You introduce two ball bearings at the upper right-hand corner. One travels by path AB and the other by path CD . Which will arrive first at the lower left-hand corner?

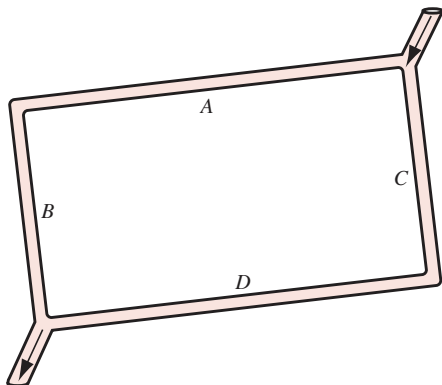


FIGURE 4-24. Question 8.

9. In broad jumping, sometimes called long jumping, does it matter how high you jump? What factors determine the span of the jump?
10. Why doesn't the electron in the beam from an electron gun fall as much because of gravity as a water molecule in the stream from a hose? Assume horizontal motion initially in each case.
11. At what point or points in its path does a projectile have its minimum speed? Its maximum?
12. Figure 4-25 shows the path followed by a NASA airplane in a run designed to simulate low-gravity conditions for a short period of time. Make an argument to show that, if the plane follows a particular parabolic path, the passengers will experience weightlessness.

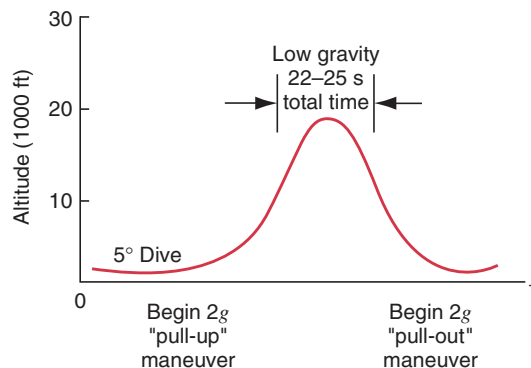


FIGURE 4-25. Question 12.

13. A shot-putter heaves a shot from above ground level. The launch angle that will produce the longest range is less than 45° ; that is, a flatter trajectory has a longer range. Explain why.
14. Consider a projectile at the top of its trajectory. (a) What is its speed in terms of v_0 and ϕ_0 ? (b) What is its acceleration? (c) How is the direction of its acceleration related to that of its velocity?
15. Trajectories are shown in Fig. 4-26 for three kicked footballs. Pick the trajectory for which (a) the time of flight is least, (b) the vertical velocity component at launch is greatest, (c) the horizontal velocity component at launch is greatest, and (d) the launch speed is least. Ignore air resistance.

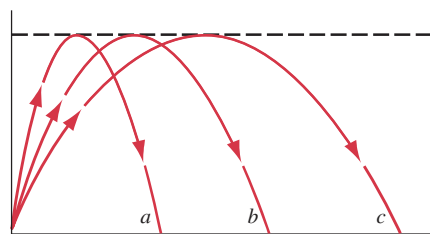


FIGURE 4-26. Question 15.

16. A rifle is bore-sighted with its barrel horizontal. Show that, at the same range, it will shoot too high when shoot-